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PROPER COVERS FOR LEFT *U*-AMPLE SEMIGROUPS

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Abstract

A left U-semiadequate semigroup is a left U-semiabundant semigroup whose projections commute. Let (S, U) be a left U-semiadequate semigroup. It is the fact that each $\widetilde{\mathcal{R}}^U$-class of (S, U) contains a unique projection. For an element a of (S, U), the projection in the $\widetilde{\mathcal{R}}^U$-class containing a is denoted by $a^{\dagger}$. If (S, U) satisfying left ample condition (AL), then we say that (S, U) is a left U-ample semigroup. In this paper, we introduce the concept of a proper cover of a left U-ample semigroup and prove that any proper cover for a left U-ample semigroup is a proper cover over a monoid. A structure theorem of proper covers for left U-ample semigroups is obtained. This theorem generalizes the result of Guo-Xie for left type A semigroups.

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1. Introduction

Let S be a semigroup and $E(S)$ be the set of all idempotents of S. Consider a non-empty subset $E \subseteq E(S)$. Let E be a commutative subsemigroup, that is, a subsemilattice of S, and let ${}^{\dagger} : S \to E$ be a unary operation. From [1], S is called left ample, if the left ample condition

$$ae = (ae)^{\dagger} a \text{ for all } a \in S \text{ and } e \in E, \tag{AL}$$

hold. Dually, we can define the right ample semigroups. An ample semigroup is one which is both left and right ample. Note that any inverse semigroup is ample. Left ample semigroups used to be known as left type A semigroups, which studied by many scholars, such as Fountain [2]; Fountain-Gomes [4]; Armstrong [5]; Guo-Xie [6] and many others.

Classes of semigroups more general than (left) ample semigroups but which satisfy one or both of the ample conditions have been widely studied. We can see from [8, 9, 10, 11, 12, 13, 14, 15, 16].

The left ample condition is easily seem to hold in a class of right PP monoids [3]; indeed, they are just left ample monoids with central idempotents, and so these results suggest that the structure theory for inverse semigroups should inform the study left ample semigroups. There are three main approaches for investigating the structure of inverse semigroups. One approach is via McAlister's theory of E-unitary covers and the P-theorem.

Covers for semigroups early occurs in the work of McAlister [18], [19] on inverse semigroups. McAlister's work has been extended in various ways by many authors, including Szendrei, Takizawa, Trotter, Fountain, Alemida, Pin and Weil.

Consider a non-empty subset $U \subseteq E(S)$, namely, the set of projections of S. Then we define a relation $\widetilde{\mathcal{R}}^U$ on S by $a\widetilde{\mathcal{R}}^U b$ if and only if a and b have the same set of left identities in U. That is, $U_a^l = U_b^l$, where $U_a^l = \{u \in U \mid ua = a\}$. The relation $\widetilde{\mathcal{L}}^U$ is defined dually. It can be easily verified that $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}^U$. Clearly, the relation $\widetilde{\mathcal{R}}^U$ on S is a natural generalization of the well-known Greens relation $\mathcal{R}$ and also the Greens star relation $\mathcal{R}^*$ adopted by Fountain, in studying abundant semigroups [1]. Recall that a semigroup S is said to be left U-semiabundant [2, 3] if every $\widetilde{\mathcal{R}}^U$-class contains some projections of U, denoted by (S, U). In the recent years, special attention has been concentrated on the set of projections U of a semigroup S instead of considering the whole set of idempotents $E(S)$ of S (see [4]), in particular, the U-semiabundant semigroups and some of its special subclasses have been extensively studied.

The aim of this paper is to generalize the structure of proper covers for a left type A semigroup in [6] to a class of U-semiabundant semigroups which here we call left U-ample semigroups. First, we introduce the concept of a proper cover of a left U-ample semigroup and then prove that any proper cover for a left U-ample semigroup is a proper cover over a monoid. A structure theorem of proper covers for left U-ample semigroups is obtained.

We use the notation and terminology of [20] and [6].

2. Preliminaries

The object in this section is to introduce the concept of proper covers for left U-ample semigroups. Throughout this paper, a left U-semiabundant semigroups is denoted by (S, U) unless it is specified otherwise. Before starting our approach, we recall some terminology, notations, and results.

From [16], the following lemma gives a basic property of the relation $\widetilde{\mathcal{R}}^U$ on a left U-semiabundant semigroups (S, U).

Lemma 2.1 ([16]). *Let (S, U) be a left U-semiabundant semigroup and e be an element of U, then the following are equivalent for $a \in (S, U)$:*

(1) $a\widetilde{\mathcal{R}}^U e$;

(2) $ea = a$ *and for all* $x \in U$, $xa = a$ *implies* $xe = e$.

In a left U-semiabundant semigroups (S, U), each $\widetilde{\mathcal{R}}^U$-class contains some projections of U. A left U-semiadequate semigroup is left U-semiabundant whose projections commute.

Lemma 2.2. *Let (S, U) be a left U-semiadequate semigroup and e, f be elements of U. If $e\widetilde{\mathcal{R}}^U f$, then $e = f$.*

Proof. If $e\widetilde{\mathcal{R}}^U f$, by Lemma 2.1, we have $e = fe = ef = f$. □

From Lemma 2.2, we see that if (S, U) is a left U-semiadequate semigroup, then each $\widetilde{\mathcal{R}}^U$-class of (S, U) contains a unique projection. For an element a of such a semigroup, the projection in the $\widetilde{\mathcal{R}}^U$-class containing a is denoted by $a^{\dagger}$.

A left U-semiadequate semigroup (S, U) is called left U-ample, if the left ample condition (AL) hold. That is,

$$ae = (ae)^{\dagger} a \text{ for all } a \in (S, U) \text{ and } e \in U. \qquad \text{(AL)}$$

Remark. (1) Dually, we can define the right U-ample semigroups. An U-ample semigroup is one which is both left and right U-ample.

(2) Since the relation $\widetilde{\mathcal{R}}^U$ is a natural generalization of the Green's star relation, the left U-ample semigroups can be think as a generalization of the left type A semigroups.

Next, we introduce a congruence σ on a left U-ample semigroups (S, U), which has the property that $(S, U)/\sigma$ is a monoid and $U \subseteq (1_{(S,U)/\sigma})(\sigma^{\natural})^{-1}$.

Lemma 2.3. *Let* (S, U) *be a left* U*-ample semigroup, a relation* σ *on* (S, U) *is given by the rule*

$$a\sigma b \Leftrightarrow \textit{exist } e \in U \textit{ such that } ea = eb.$$

Then σ *is a congruence,* $(S, U)/\sigma$ *is monoid and* $U \subseteq (1_{(S,U)/\sigma})(\sigma^{\natural})^{-1}$. *If* τ *is a monoid congruence on* (S, U) *such that* $U \subseteq (1_{(S,U)/\tau})(\tau^{\natural})^{-1}$, *then* $\sigma \subseteq \tau$. *In particular,* σ *is the minimum monoid congruence on* (S, U) *such that* $U \subseteq (1_{(S,U)/\sigma})(\sigma^{\natural})^{-1}$.

Proof. It is clear that σ is an equivalence and right compatible.

[Since $ea = ea$ for all $a \in (S, U)$ and $e \in U$, we have $a\sigma a$. Hence σ is reflexive.

If $a\sigma b$, then exist $e \in U$ such that $ea = eb$. Also, we have $eb = ea$ and so $b\sigma a$. Hence σ is symmetric.

If $a\sigma b$ and $b\sigma c$, then exist $e, f \in U$ such that $ea = eb$ and $fb = fc$. Also, we have $fea = feb$ and $efb = efc$. Since $ef = fe$, we have $fea = feb = efb = efc = fec$ and so $a\sigma c$. Hence σ is transitive.

Now, we have already proved that σ is an equivalence. Let $a, b, c \in (S, U)$ and $a\sigma b$, then

$$\begin{aligned} a\sigma b &\Rightarrow \text{exist } e \in U \text{ such that } ea = eb \\ &\Rightarrow eac = ebc \\ &\Rightarrow ac\sigma bc. \end{aligned}$$

Hence σ is right compatible as required.]

To prove that σ is a congruence, it remain to show that σ is left compatible. Let $a, b, c \in S$ and $a\sigma b$, then

$$\begin{aligned} a\sigma b &\Rightarrow \text{exist } e \in U \text{ such that } ea = eb \\ &\Rightarrow cea = ceb \\ &\Rightarrow (ce)^{\dagger} ca = (ce)^{\dagger} cb \qquad \text{((AL))} \\ &\Rightarrow ca\sigma cb. \end{aligned}$$

Hence σ is left compatible as required. Since projections commute, we have $e\sigma f$ for any $e, f \in U$. Let $e \in U$, we claim that $e\sigma$ is an identity of $(S, U)/\sigma$. This is because for any $a\sigma \in (S, U)/\sigma$, we have

$$e\sigma.a\sigma = a^{\dagger}\sigma.a\sigma = (a^{\dagger}a)\sigma = a\sigma,$$

and

$$ae = (ae)^{\dagger}a \Rightarrow (ae)^{\dagger}ae = (ae)^{\dagger}a \Rightarrow ae\sigma a \Rightarrow ae\sigma = a\sigma \Rightarrow a\sigma.e\sigma = a\sigma.$$

Hence $U \subseteq (1_{(S,U)/\sigma})(\sigma^{\natural})^{-1}$. Up to now, we have already established that σ is a congruence, $(S, U)/\sigma$ is a monoid and $U \subseteq (1_{(S,U)/\sigma})(\sigma^{\natural})^{-1}$. Now, it remain to show that σ is the least monoid congruence such that $U \subseteq (1_{(S,U)/\sigma})(\sigma^{\natural})^{-1}$. Let τ be a monoid congruence with $U \subseteq (1_{(S,U)/\tau})(\tau^{\natural})^{-1}$. It is easy to see that for any $e \in U$, $e\tau = 1_{(S,U)/\tau}$. [Since $U \subseteq (1_{(S,U)/\tau})(\tau^{\natural})^{-1}$, we have $e\tau = 1_{(S,U)/\tau}$.]

Now

$$\begin{aligned} a\sigma b &\Rightarrow \text{exist } f \in U \text{ such that } fa = fb \\ &\Rightarrow f\tau.a\tau = f\tau.a\tau \\ &\Rightarrow a\tau = b\tau \qquad (\text{since } f\tau = 1_{(S,U)/\tau}) \\ &\Rightarrow a\tau b. \end{aligned}$$

Hence $\sigma \subseteq \tau$ and so σ is the minimum monoid congruence on (S, U) such that $U \subseteq (1_{(S,U)/\sigma})(\sigma^{\natural})^{-1}$ as required. □

Let (S, U) and (T, V) are two left U-semiabundant semigroups. Similar to the definition of $\mathcal{L}^*$-homomorphism in [3], a homomorphism ϕ from (S, U) to (T, V) is called $\widetilde{\mathcal{R}}^U$-homomorphism if for all $a, b \in S$, $a\phi = b\phi$ implies $a\widetilde{\mathcal{R}}^U b$ and $\phi|_U : U \to V$.

Definition 2.1. Let (T, V) be a left U-ample semigroup, some definitions on (T, V) are as follow:

(D1) If $\widetilde{\mathcal{R}}^U \cap \sigma = 1_{(T,V)}$, then we call (T, V) is proper.

(D2) If (T, V) is proper, ϕ is a $\widetilde{\mathcal{R}}^U$-homomorphism from (T, V) onto a left U-ample semigroup (S, U) and for any $e \in U$, there exist $f \in V$ such that $f\phi = e$, then we call (T, V) is a proper cover for (S, U).

(D3) If (T, V) is a proper cover for (S, U), M is a monoid, $(T, V)/\sigma \cong M$ and $V \subseteq 1_M \alpha^{-1}(\sigma^{\natural})^{-1}$, where α is an isomorphism from $(T, V)/\sigma$ onto M, then we call (T, V) is a proper cover for (S, U) (over M).

From [20], a subset A of a semigroup S is called left unitary if for all $a \in A$ and $s \in S$, $as \in A$ implies $s \in A$. Dually, we can define right unitary. S is called unitary if it is both left unitary and right unitary.

Lemma 2.4. *Let (S, U) be a left U-ample semigroup, if (S, U) is proper, then it is U-unitary.*

Proof. Let $e \in U$ and $a \in (S, U)$. On the one hand, if $ae \in U$, then

$$(ae)^{\dagger}.ae = ae = (ae)^{\dagger}a = (ae)^{\dagger}.a.$$

Hence, $ae\sigma a$. Since $ae \in U$ and so $a^{\dagger}\sigma ae\sigma a$. Then we have $a(\widetilde{\mathcal{R}}^U \cap \sigma)a^{\dagger}$. Since (S, U) is proper, we have $\widetilde{\mathcal{R}}^U \cap \sigma = 1_{(S,U)}$ and so $a = a^{\dagger} \in U$.

On the other hand, if $ea \in U$, then $a = a^{\dagger} a\sigma e a\sigma a^{\dagger}$. Hence $a(\widetilde{\mathcal{R}}^{U} \cap \sigma)a^{\dagger}$. Since (S, U) is proper, we have $\widetilde{\mathcal{R}}^{U} \cap \sigma = 1_{(S,U)}$ and so $a = a^{\dagger} \in U$. □

3. The Main Result

In this section, we will show that any proper cover for a left U-ample semigroup is a proper cover over a monoid. A structure theorem of proper covers for left U-ample semigroup is obtained.

Definition 3.1. Let (S, U) be a left U-ample semigroup, M be a monoid. A surjective relational morphism θ from M to (S, U) is a mapping $\theta : M \to 2^{(S,U)}$ such that

(A1) $m\theta \neq \emptyset$ for all $m \in M$;

(A2) $m_1\theta . m_2\theta \subseteq (m_1 m_2)\theta$ for all $m_1, m_2 \in M$;

(A3) $\bigcup_{m \in M} m\theta = (S, U)$;

(A4) $1\theta = U$;

(A5) $|\widetilde{\mathcal{R}}_a^{U} \cap m\theta| \leq 1$ for all $m \in M$, $a \in (S, U)$;

(A6) $m\theta \subseteq a\sigma$ for all $m \in M$, $a \in m\theta$.

Theorem 3.1. *Let (S, U) be a left U-ample semigroup, M be a monoid, and θ be a surjective relational morphism from M to (S, U). Let*

$$T = \{(s, m) \in (S, U) \times M \mid s \in m\theta\},$$

and define a multiplication on T by

$$(s_1, m_1)(s_2, m_2) = (s_1 s_2, m_1 m_2).$$

Then T is a semigroup and

(1) $V = \{(e, 1) | e \in U\}$ *is a subset of* $E(T)$ *and* $V \cong U$;

(2) *for all* $a, b \in S$, $g, h \in M$, $(a, g)\widetilde{\mathcal{R}}^V(b, h) \Leftrightarrow a\widetilde{\mathcal{R}}^U b$;

(3) (T, V) *is a left V-ample semigroup*;

(4) *for all* $(a, g), (b, h) \in (T, V)$, $(a, g)\sigma_{(T,V)}(b, h) \Leftrightarrow a\sigma_{(S,U)}b, g = h$.

Proof. Let T be as in the statement of the theorem. It is clearly that T is a semigroup. Now we proof the rest.

(1) Since θ is a surjective relational morphism, by (A4), we have V is a subsemigroup of T and $T \subseteq E(T)$. Then it follows that $V \cong U$.

(2) It is benefit for us to prove the following useful lemma.

Lemma 3.1. *Let* $(a, g) \in (T, V)$, *then* $(a, g)\widetilde{\mathcal{R}}^V(a^\dagger, 1)$.

Proof. Let $(a, g) \in (T, V)$. By (1), we have $(a^\dagger, 1) \in V$. It is clear that $(a^\dagger, 1)(a, g) = (a, g)$. Now, for any $(f, 1) \in V$ if $(f, 1)(a, g) = (a, g)$, then

$$\begin{aligned}(f, 1)(a, g) = (a, g) &\Rightarrow f.a = a \text{ and } 1.g = g\\ &\Rightarrow f.a^\dagger = a^\dagger \qquad (\text{since } a\widetilde{\mathcal{R}}^U a^\dagger)\\ &\Rightarrow (f, 1)(a^\dagger, 1) = (a^\dagger, 1).\end{aligned}$$

By Lemma 2.1, we have $(a, g)\widetilde{\mathcal{R}}^V(a^\dagger, 1)$ as required. □

Returning now to the main proof. Let $a, b \in (S, U)$, $g, h \in M$. On the one hand, if $(a, g)\widetilde{\mathcal{R}}^V(b, h)$, by Lemma 3.1, we have

$$(a^\dagger, 1)\widetilde{\mathcal{R}}^V(a, g)\widetilde{\mathcal{R}}^V(b, h)\widetilde{\mathcal{R}}^V(b^\dagger, 1),$$

and so $(a^\dagger, 1)\widetilde{\mathcal{R}}^V(b^\dagger, 1)$. By Lemma 2.2, it follows that $(a^\dagger, 1) = (b^\dagger, 1)$ and so $a^\dagger = b^\dagger$. Hence $a\widetilde{\mathcal{R}}^U b$ as required.

On the other hand, if $a\widetilde{\mathcal{R}}^U b$, then $a^{\dagger}\widetilde{\mathcal{R}}^U b^{\dagger}$ and so $a^{\dagger} = b^{\dagger}$. By Lemma 3.1, we have

$$(a, g)\widetilde{\mathcal{R}}^V(a^{\dagger}, 1) = (b^{\dagger}, 1)\widetilde{\mathcal{R}}^V(b, h).$$

Hence $(a, g)\widetilde{\mathcal{R}}^V(b, h)$ as required.

(3) From (1) and (2), we have (T, V) is left V-semiabundant and projections commute and so is left V-semiadequate. Let $(e, 1) \in V$ and $(a, g) \in T$, where $e \in U$. Since (S, U) is a left U-ample semigroup, we have

$$(a, g)(e, 1) = (ae, g) = ((ae)^{\dagger}a, g) = ((ae)^{\dagger}, 1)(a, g) = [(a, g)(e, 1)]^{\dagger}(a, g).$$

Hence (T, V) satisfies the left ample condition (AL) and so is a left U-ample semigroup.

(4) Let $(a, g)(b, h) \in (T, V)$, if $(a, g)\sigma_{(T,V)}(b, h)$, there exist $(e, 1) \in V$ such that $(e, 1)(a, g) = (e, 1)(b, h)$ and so $(ea, g) = (eb, h)$. That is, $ea = eb$ and $g = h$. Hence $a\sigma_{(S,U)}b$ and $g = h$. Conversely, if $a\sigma_{(S,U)}b$ and $g = h$, then exist $e \in U$ such that $ea = eb$ and so $(e, 1)(a, g) = (ea, g) = (eb, h) = (e, 1)(b, h)$. Since $(e, 1) \in V$, we have $(a, g)\sigma_{(T,V)}(b, h)$. $\square$

Theorem 3.2. *Let (S, U) be a left U-ample semigroup, M be a monoid, and θ be a surjective relational morphism from M to (S, U). Let*

$$T = \{(s, m) \in (S, U) \times M \mid s \in m\theta\},$$

and define a multiplication on T by

$$(s_1, m_1)(s_2, m_2) = (s_1s_2, m_1m_2).$$

Let $V = \{(e, 1) \mid e \in U\}$, then (T, V) is a proper cover of (S, U) over M. Conversely, any proper cover of (S, U) can be constructed in this way.

Proof. From Theorem 3.1, we have (T, V) is a left V-ample semigroup. Let $(a, g)(b, h) \in (T, V)$ and $(a, g)(\widetilde{\mathcal{R}}^V \cap \sigma_{(T,V)})(b, h)$. By Theorem 3.1 (2), (4), we have $a\widetilde{\mathcal{R}}^U b$ and $g = h$ and so $a, b \in h\theta = g\theta$. Since θ is a surjective relational morphism, by (A5), we have $a = b$. Hence $(a, g) = (b, h)$. That is $\widetilde{\mathcal{R}}^V \cap \sigma_{(T,V)} = 1_{(T,V)}$. Thus (T, V) is proper.

A mapping β from (T, V) to (S, U) is defined by the rule as follow:

$$\beta : (T, V) \to (S, U), (a, g) \mapsto a.$$

It is clear that β is a surjective homomorphism and $\beta|_V : V \to U$. On the one hand, if $[(a, g)]\beta = [(b, h)]\beta$ for $(a, g)(b, h) \in (T, V)$, then $a = b$ and so $a\widetilde{\mathcal{R}}^U b$. By Theorem 3.1 (2), we have $(a, g)\widetilde{\mathcal{R}}^V(b, h)$. Hence β is $\widetilde{\mathcal{R}}^V$-homomorphism from (T, V) onto (S, U). On the other hand, for any $e \in U$, by Theorem 3.1 (1), we have $(e, 1) \in V$ and so $[(e, 1)]\beta = e$. Thus (T, V) is a proper cover for (S, U).

Since $\sigma_{(T,V)}$ is the least monoid congruence with $V \subseteq (1_{(T,V)/\sigma_{(T,V)}})(\sigma^{\natural}_{(T,V)})^{-1}$. A mapping α from $(T, V)/\sigma_{(T,V)}$ to M is defined by the rule as follow:

$$\alpha : (T, V)/\sigma_{(T,V)} \to M, (a, g)\sigma_{(T,V)} \mapsto g.$$

We claim that α is a one-one mapping. Let $(a, g)\sigma_{(T,V)}, (b, h)\sigma_{(T,V)} \in (T, V)/\sigma_{(T,V)}$ and $(a, g)\sigma_{(T,V)} = (b, h)\sigma_{(T,V)}$. Since $(a, g)\sigma_{(T,V)}(b, h)$, by Theorem 3.1 (4), we have $a\sigma_{(S,U)}b$ and $g = h$. That is $[(a, g)\sigma_{(T,V)}]\alpha = g = h = [(b, h)\sigma_{(T,V)}]\alpha$, hence α is a mapping. It is clear that α is surjective. If $[(a, g)\sigma_{(T,V)}]\alpha = [(b, h)\sigma_{(T,V)}]\alpha$ for $(a, g)\sigma_{(T,V)}$, $(b, h)\sigma_{(T,V)} \in (T, V)/\sigma_{(T,V)}$. Then $g = h$ and so $a, b \in g\theta$. By (A6), we have $a\sigma_{(S,U)}b$. So by Theorem 3.1 (4), we have $(a, g)\sigma_{(T,V)}(b, h)$. Thus α is one-one as required. On the one hand, let $(a, g)\sigma_{(T,V)}, (b, h)\sigma_{(T,V)} \in (T, V)/\sigma_{(T,V)}$, since

$$[(a, g)\sigma_{(T,V)}, (b, h)\sigma_{(T,V)}]\alpha = [(ab, gh)\sigma_{(T,V)}]\alpha = gh$$

$$= [(a, g)\sigma_{(T,V)}]\alpha.[(b, h)\sigma_{(T,V)}]\alpha.$$

Hence α is an isomorphic. On the other hand, since $V \subseteq (1_{(T, V)/\sigma_{(T,V)}})(\sigma^{\natural}_{(T,V)})^{-1}$ and $(T, V)/\sigma_{(T,V)} \cong M$, we have $V \subseteq 1_M \alpha^{-1}(\sigma^{\natural}_{(T,V)})^{-1}$. Thus (T, V) is a proper cover for (S, U) over M. Up to now, we have already established the first statement in this theorem.

Conversely, let (T, V) be a proper cover for (S, U). Then there is a $\widetilde{\mathcal{R}}^V$-homomorphism ϕ from (T, V) onto (S, U) satisfying for any $e \in U$, there exist $f \in V$ such that $f\phi = e$. Let $M = (T, V)/\sigma_{(T,V)}$, by Lemma 2.3, M is a monoid with $V \subseteq 1_M(\sigma^{\natural}_{(T,V)})^{-1}$.

A relation morphism θ from M to (S, U) is defined by the rule as follow:

$$\theta : M \to 2^{(S,U)},\ g \mapsto g\theta,$$

for any $g \in M$, $g\theta = \{s \in (S, U) | \text{exist } t \in (T, V),\ s = t\phi,\ t\sigma_{(T, V)} = g\}$. It remain to prove that θ is a surjective relational morphism and $(T, V) \cong (T', V')$, where

$$T' = \{(s, g) \in (S, U) \times M | s \in g\theta\},$$

$$V' = \{(e, 1) \in (S, U) \times M | e \in 1\theta\}.$$

Let $g \in M$, since the natural morphism $\sigma^{\natural}_{(T,V)} : (T, V) \to (T, V)/\sigma_{(T,V)} = M$ is surjective, we have $g\theta \neq \emptyset$ and so θ satisfies condition (A1).

Let $g, h \in M$, $s_1 \in g\theta$, $s_2 \in h\theta$, then exist $u, v \in (T, V)$ such that

$$s_1 = u\phi,\ u\sigma_{(T,V)} = g,\ s_2 = v\phi,\ v\sigma_{(T,V)} = h.$$

Then we have $s_1 s_2 = (uv)\phi$, $(uv)\sigma_{(T,V)} = gh$ and so $s_1 s_2 \in (gh)\theta$. Hence θ satisfies condition (A2).

It is clear that $\bigcup_{g\in M} g\theta = (S, U)$. Hence θ satisfies condition (A3).

Let $s \in 1\theta$, then exist $t \in (T, V)$ such that $s = t\phi$, $t\sigma_{(T,V)} = 1$. Note that $t^{\dagger}\sigma_{(T,V)} = 1$, we have $t(\widetilde{\mathcal{R}}^U \cap \sigma_{(T,V)})t^{\dagger}$. Since (T, V) is proper, we have $t = t^{\dagger} \in V$. Hence $1\theta \subseteq U$. Conversely, if $e \in U$, then there exist $f \in V$ such that $e = f\phi$. Since $f\sigma_{(T,V)} = 1$, we have $e \in 1\theta$ and so $U \subseteq 1\theta$. Hence θ satisfies condition (A4).

Now, we prove that $(T, V) \cong (T', V')$ first. A mapping ψ from (T, V) to (T', V') is defined by the rule as follow:

$$\psi : (T, V) \to (T', V'),\ t \mapsto (t\phi, t\sigma_{(T,V)}),$$

it is clear that ψ is a surjective morphism. Let $t, u \in (T, V)$ and $(t\phi, t\sigma_{(T,V)}) = t\psi = u\psi = (u\phi, u\sigma_{(T,V)})$. Then, we have $t\phi = u\phi$, $t\sigma_{(T,V)} = u\sigma_{(T,V)}$. Since ϕ is a $\widetilde{\mathcal{R}}^U$-homomorphism, we have $t\widetilde{\mathcal{R}}^V u$ and so $t(\widetilde{\mathcal{R}}^V \cap \sigma_{(T,V)})u$. Since (T, V) is proper, we have $t = u$. Hence ψ is an isomorphism. Finally, since $\psi|_V = V'$, we have $V \cong V'$ and so $(T, V) \cong (T', V')$. That is (T', V') is proper.

To show that θ satisfies condition (A5). Let $s, s' \in g\theta$ and $s\widetilde{\mathcal{R}}^U s'$, then $(s, g), (s', g) \in (T', V')$. By Theorem 3.1 (2), we have $(s, g)\widetilde{\mathcal{R}}^{V'}(s', g)$. Since ψ is a isomorphism, there exist $t, t' \in (T, V)$ such that

$$t\psi = (t\phi, t\sigma_{(T,V)}) = (s, g),\ t'\psi = (t'\phi, t'\sigma_{(T,V)}) = (s', g).$$

Then $t\sigma_{(T,V)}t'$ and so there exist $e \in V$ such that $et = et'$. Then

$$e\psi.(s,\ g) = e\psi.t\psi = (et)\psi = (et')\psi = e\psi.t'\psi = e\psi.(s',\ g),$$

and so $(s',\ g)\sigma_{(T',V')}(s,\ g)$. Then we have $(s',\ g)(\widetilde{\mathcal{R}}^{V'} \cap \sigma_{(T',V')})(s,\ g)$. Since T' is proper, we have $(s',\ g) = (s,\ g)$ and so $s = s'$. Hence θ satisfies condition (A5) as required.

Finally, we show that θ satisfies condition (A6). Let $s, t \in (S, U)$ and $m \in M$ such that $s, t \in m\theta$, then we have $(s, m), (t, m) \in (T', V')$. Similar to the prove of θ satisfies condition (A5), we have $(s, m)\sigma_{(T',V')}(t, m)$. Similar to the prove of Theorem 3.1 (4), we have $s\sigma_{(S,U)}t$. Hence θ satisfies condition (A6) as required. $\square$

Remark. (1) By Theorem 3.1 and the direct part proof of the Theorem 3.2, we have the following diagram:

$$\begin{tikzcd}
V \arrow[r] \arrow[d, "\beta"'] & (T,V) \arrow[d, "\beta"'] \arrow[dr, "\sigma^{\natural}_{(T,V)}\circ\alpha"] & \\
U \arrow[r] & (S,U) & M
\end{tikzcd}$$

where β is a $\widetilde{\mathcal{R}}^U$-homomorphism from (T, V) onto (S, U), $\sigma^{\natural}_{(T,V)}$ is a natural morphism and $\beta|_V$ is an isomorphism.

(2) From the converse part proof of Theorem 3.2, we have the following diagram:

$$\begin{tikzcd}
V \arrow[r] \arrow[d, "\phi"'] & (T,V) \arrow[d, "\phi"'] \arrow[dr, "\sigma^{\natural}_{(T,V)}"] & \\
U \arrow[r] & (S,U) & M
\end{tikzcd}$$

where $M = (T, V)/\sigma_{(T,V)}$, ϕ is a $\widetilde{\mathcal{R}}^U$-homomorphism from (T, V) onto (S, U), $\sigma^{\natural}_{(T,V)}$ is a natural morphism and $\phi|_V$ is an isomorphism.

(3) Since the relation $\widetilde{\mathcal{R}}^U$ is a natural generalization of the Green's star relation, the left U-ample semigroups can be think as a generalization of left type A semigroups. From this point, this theorem generalizes the result of Guo-Xie [6] for left type A semigroups.

The following result is immediate from the Lemma 2.4 and the Theorem 3.2.

Corollary 3.1. *A left U-ample semigroup has a U-unitary proper cover over a monoid.*

Dually, we have the following results:

Corollary 3.2. *A* [*right*] *U-ample semigroup has a U-unitary proper cover over a monoid.*

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